Journal of Computational and Applied Mathematics 289 (2015) 322–330

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

First order partial differential equations with time delay and retardation of a state variable

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ARTICLE INFO

Article history:
Received 6 September 2014
Received in revised form 1 December 2014

MSC:
65Q20
65M06

Keywords:
Difference scheme
Partial delay differential equation
Retardation
Time delay

ABSTRACT

We construct a finite difference scheme for the numerical solution of a first order partial differential equation with a time delay and retardation of a state variable. Such equations are used to model the dynamics of structured cell populations when age and maturity level are taken into account. For the supplied difference schemes the order of approximation, stability and convergence order are studied. We illustrate the obtained results with a test example.

1. Introduction

First order partial differential equations with time delay and retardation of a state variable, also known as advection equations with distributed parameters, arise in the modeling of dynamics of populations structured with respect to the cell size, the age of specimen, maturation level etc. [1–3]. The authors in [2] note that the dynamics are not only dependent on the behavior of the cell population numbers some time in the past (time delayed effects), but also that the population behavior at a given maturation level is dependent on the behavior at a previous maturation level (nonlocal effects). Thus, this important biological problem leads in a rather natural fashion to a complex mathematical problem involving delayed nonlocal dynamics described by a nonlinear advection equation. When diffusion is more dominant, such as in elasto-plasticity and in the theory of reactive contaminant transport, time delay can also occur and can be modeled through a convolution term, see e.g. [4,5].

The qualitative theory of partial functional differential equations (PFDE) in general form is elaborated quite well (see, for example, [6] and references therein). Papers which deal with an advection equation with time delay and retardation of a state variable and its applications in cell dynamics usual consider questions of existence, uniqueness and global stability. As a general rule the equation is rewritten as a linear evolution problem in a Banach space and results are formulated in terms of a strongly continuous semigroup of bounded linear operators. Particular systems were analyzed numerically [7–9]. Nevertheless, numerical methods for the equation in general form were not constructed and theorems of its convergence were not formulated.

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http://dx.doi.org/10.1016/j.cam.2014.12.032

Since in most cases one cannot find the explicit solution of PFDE, the elaboration, substantiation, and program realization of numerical methods for these equations are of essential interest. Below we review some approaches to numerically solve such equations.

Method of lines [10–12] reduces PFDE to a system of differential equations with time delay in ordinal derivatives which could be solved by special methods [13–15]; unfortunately after discretization with respect to state variables a stiff system appears. Implicit difference methods for first order PFDE [16–18] allow to avoid this stiffness by an appropriate choice of the step size. However to obtain a solution on each next time layer one must solve high-dimensional nonlinear systems.

Especially effective difference schemes for PFDE of parabolic and hyperbolic type were elaborated in [19–22]. The main idea in these works is a separation principle that consists of distinguishing finite and infinite dimensional components in the structure of PFDE. To take into account the time delay effect, interpolation and extrapolation of discrete prehistory is used. This extrapolation is needed for the realization of implicit methods and allows the authors to avoid the necessity of solving nonlinear systems.

The present paper continues the investigation initiated in [23]. Our approach is close to [19] and is based on a combination of the stability verification methods for two-layer difference schemes [24] and the separation principle mentioned above.

Let $\tau > 0$ and consider the following advection equation with aftereffect and retardation of a state variable

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = f(x, t, u(x, t), u_{(t)}(\alpha x, \cdot)), \tag{1a}$$

where $x \in [0, X]$ is a state and $t \in [t_0, \theta]$ is time; $u(x, t)$ is an unknown function; $u_{(t)}(\alpha x, \cdot) = \{u(\alpha x, t + \xi), \ -\tau \le \xi < 0\}$ is a prehistory-function of the unknown function to the moment t which also involves biasing in the state variable, $\alpha \in (0, 1)$ is the constant of biasing, and $a > 0$ is a constant. Together with the advection equation we have the following initial condition

$$u(t, x) = \varphi(x, t), \quad x \in [0, X], \ t \in [t_0 - \tau, t_0] \tag{1b}$$

and the boundary condition

$$u(0, t) = g(t), \quad t \in [t_0, \theta]. \tag{1c}$$

We adopt the compatibility condition $g(t_0) = \varphi(0, t_0)$. Questions of the existence and uniqueness of a solution to the stated boundary value problem (1) were considered in [6] and we assume that the functional f and functions φ and g are such that problem has a unique solution.

We denote by $\mathcal{Q} = \mathcal{Q}[-\tau, 0)$ the set of functions $u(\xi)$ that are piecewise continuous on $[-\tau, 0)$ with a finite number of points of discontinuity of the first kind and right continuous at the points of discontinuity. We define a norm on $\mathcal{Q}$ by $\|u\|_{\mathcal{Q}} = \sup_{\xi\in[-\tau,0)} |u(\xi)|$. We additionally assume that the functional $f(x, t, u, v(\cdot))$ is given on $[0, X] \times [t_0, \theta] \times \mathbb{R} \times \mathcal{Q}$ and is Lipschitz in the last two arguments.

2. The difference scheme

We consider an equidistant partition of $[0, X]$ into parts of size $h = X/N$ and split the time frame $[t_0, \theta]$ into parts with size $\Delta = (\theta - t_0)/M$. We obtain the uniform grid $\{x_i, t_j\}_{j=0}^{M}{}_{i=0}^{N}N$, where $t_j = t_0 + j\Delta$, $j = 0, \ldots, M$, and $x_i = ih$, $i = 0, \ldots, N$. Denote by u_j^i approximations of the functions $u(x_i, t_j)$, $i = 0, \ldots, N$, $j = 0, \ldots, M$, at the nodes. Without loss of generality and to simplify the narration we assume that the value $\tau/\Delta = m$ is a natural number.

Since functional $f(x_i, t_j, u(x_i, t_j), u_{t_j}(\alpha x_i, \cdot))$ may depend on values of the function u between grid nodes, interpolation may be needed. For every fixed node (x_i, t_j) and time delay $\xi \in [-\tau, 0)$ there are only three possibilities:

(1) $t_j + \xi \le t_0$: interpolation is not needed, we use the initial function, $u(\alpha x_i, t_j + \xi) = \varphi(\alpha x_i, t_j + \xi)$;
(2) $t_j + \xi > t_0$ and $\alpha x_i = 0$ (so $x_i = 0$): interpolation is not needed, we use the boundary function, $u(\alpha x_i, t_j + \xi) = u(0, t_j + \xi) = g(t_j + \xi)$;
(3) $t_j + \xi > t_0$ and $\alpha x_i > x_0$: we use the interpolation as described below.

For every fixed node (x_i, t_j), $i = 1, \ldots, N$, $j = 1, \ldots, M$, we introduce its discrete domain of influence

$$\left\{u_l^k\right\}_j^i = \left\{u_l^k \mid k \le \alpha i \le k + 1, \ \max\{0, j - m\} \le l \le j\right\}.$$

Definition 1 (*Interpolation Operator*)**.** A mapping I defined on the set $\mathcal{A}_j^i$ of all admissible discrete domains of influence and acting by the rule

$$I : \mathcal{A}_j^i \to \mathcal{Q}[-\min\{\tau, t_j\}, 0] : \left\{u_l^k\right\}_j^i \mapsto v^{i,j}(\cdot) = v^{i,j}(t_j + \xi)$$

is called an *interpolation operator* for the discrete history.

Let us give an example of a concrete interpolation operator, which has the properties required for the numerical method that we are going to construct. For the discrete domains of influence $\left\{u_l^k\right\}_j^i$ we set $u_l^{\alpha i} = (k+1-\alpha i)u_l^k + (\alpha i - k)u_l^{k+1},\ k \le \alpha i \le k+1$, by which we define

$$v^{i,j}(t_j+\xi) = \frac{1}{\Delta}\Big(\left(t_{l+1}-t_j-\xi\right)u_l^{\alpha i} + \left(t_j+\xi-t_l\right)u_{l+1}^{\alpha i}\Big), \quad t_l \le t_j+\xi \le t_{l+1}. \tag{2}$$

Definition 2 (*Order of Interpolation Operator*). An interpolation operator I has order (of error) p with respect to a state variable and order (of error) q with respect to time on the exact solution, if there exist constants C_1 and C_2 such that, for all $i = 1,\dots,N,\ j = 1,\dots,M$, and $t \in [\max\{0, t_j-\tau\}, t_j]$ the following inequality holds:

$$\left| I\left\{u_l^k\right\}_j^i - u(\alpha x_i, t)\right| \le C_1 \max_{\substack{\max\{0,j-m\}\le l\le j \\ k\le \alpha i\le k+1}} \left|u_l^k - u(\alpha x_i, t_l)\right| + C_2\left(h^p + \Delta^q\right).$$

For example, the operator of interpolation (2) is of second order with respect to a state variable and also with respect to time.

We consider the following family of difference schemes (parametrized by $0 \le s \le 1$), with $j = 0,\dots,M-1$:

$$\frac{u_{j+1}^1-u_j^1}{\Delta} + a\left(s\,\frac{-4u_{j+1}^0 - 2h/a(f_{j+1}^0-\dot g_{j+1}) + 4u_{j+1}^1}{2h} + (1-s)\,\frac{-4u_j^0 - 2h/a(f_j^0-\dot g_j) + 4u_j^1}{2h}\right) = f_j^1, \tag{$\mathcal{M}_s$a}$$

$$\frac{u_{j+1}^i-u_j^i}{\Delta} + a\left(s\,\frac{u_{j+1}^{i-2} - 4u_{j+1}^{i-1} + 3u_{j+1}^i}{2h} + (1-s)\,\frac{u_j^{i-2} - 4u_j^{i-1} + 3u_j^i}{2h}\right) = f_j^i, \quad i = 2,\dots,N, \tag{$\mathcal{M}_s$b}$$

with the initial condition

$$u_0^i = \varphi(x_i, t_0), \quad i = 0,\dots,N; \qquad v^{i,0}(t) = \varphi(x_i, t), \quad t < t_0,\ i = 0,\dots,N, \tag{$\mathcal{M}_s$c}$$

and boundary condition

$$u_j^0 = g_0(t_j), \quad j = 0,\dots,M. \tag{$\mathcal{M}_s$d}$$

Here $f_j^i = f\left(x_i, t_j, u_j^i, v^{i,j}(\cdot)\right)$ is the value of the functional f, calculated on an approximate solution, $v^{i,j}(\cdot)$ is the result of an interpolation, $\dot g_j = g'(t_0 + j\Delta)$. For constructing a numerical method, we additionally assume that $g(t)$ is a differentiable function.

2.1. Context and origin of the scheme $\mathcal{M}_s$

Let us put the proposed scheme $\mathcal{M}_s$ in the context of the existing finite-difference schemes for hPDEs.

So-called FTCS (Forward-Time, Central-Space) method uses the following approximation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} \xrightarrow{\text{FTCS}} \frac{u_{j+1}^i - u_j^i}{\Delta} + a\frac{u_j^{i+1}-u_j^{i-1}}{2h}$$

and therefore has a first-order in time and second-order in space. This scheme is unconditionally unstable for advection equations unless artificial viscosity is included, therefore it is not studied here.

The solution of the FTCS scheme stability problem was proposed by Lax. The main idea is based on replacing in the FTCS formula the term u_j^i with its spatial average $(u_j^{i+1} + u_j^{i-1})/2$, this guarantees the stability if the Courant condition $c \le h/\Delta$ is fulfilled [25]. The Lax scheme approximates the equation as $O(h^2 + h^2/\Delta + \Delta)$ and therefore is inconsistent. Because of the inconsistency and conditional stability, h and Δ cannot be independent. The more sophisticated Lax–Wendroff method, which could be considered as a multistep method, leads to the accuracy $O(h^2 + \Delta^2)$ and is stable under the same Courant condition. Both of these methods are widely used to solve initial problems when the initial condition $u(x, 0) = u_0,\ x \ge 0$, is defined on the semi-axis, but they are not suitable when the initial condition is defined only on the segment $[0, X]$ coupled with the boundary condition defined on the segment $[0, T]$. This is the main reason why we do not try to generalize this method in the case of hereditary systems.

Widely-known first-order upwind schemes are the particular cases of a running scheme family which is circumstantially studied in [26]. Second-order upwind schemes improve the spatial accuracy of the first-order upwind scheme by including three data points instead of just two and was the basis of the elaborated method (3). Unfortunately these schemes are not directly applicable, they must be modified near the boundary without loss of accuracy. This modification is a feature of our method.

For the advection equation with time delay grid methods were built in [27], the approach is very close to that we use in this paper. These methods are analogs of running scheme families, analogs of the Crank–Nicolson scheme and an approximation method to the middle of the square.

To conclude this subsection we explain the way in which we have obtained our numerical scheme. The derivative $\partial u/\partial t$ in (1a) is approximated by a finite difference over two nodes. For nodes (i, j), $i = 2, \dots, N$, $j = 0, \dots, M-1$, the derivative $\partial u/\partial x$ is approximated by a finite difference over three nodes on the right edge. For $i = 1$ such an approximation requires to calculate u_j^{-1}; in this case we apply the approximation over three nodes with $(0, j)$ the double node:

$$\frac{\partial u_j^1}{\partial x} \approx \frac{1}{2h}\left(-4u_j^0 - 2h\frac{\partial u_j^0}{\partial x} + 4u_j^1\right).$$

Because of (1) we have $\frac{\partial u_j^0}{\partial x} = \frac{1}{a}\left(f_j^0 - \frac{\partial u_j^0}{\partial t}\right)$, and due to (1c) we obtain $\frac{1}{a}\left(f_j^0 - \dot{g}_j\right)$.

2.2. The residual of the scheme $\mathcal{M}_s$

We call the mesh function

$$\begin{aligned}\Psi_j^1 \;=\;& \frac{u(x_1, t_{j+1}) - u(x_1, t_j)}{\Delta} + as\,\frac{-4u(x_0, t_{j+1}) - 2h/a(f_{j+1}^0 - \dot{g}_{j+1}) + 4u(x_1, t_{j+1})}{2h}\\ &+ a(1-s)\,\frac{-4u(x_0, t_j) - 2h/a(f_{j+1}^0 - \dot{g}_j) + 4u(x_1, t_j)}{2h} - \bar{f}_j^1,\end{aligned} \tag{3a}$$

$$\begin{aligned}\Psi_j^i \;=\;& \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\Delta} + as\,\frac{u(x_{i-2}, t_{j+1}) - 4u(x_{i-1}, t_{j+1}) + 3u(x_i, t_{j+1})}{2h}\\ &+ a(1-s)\,\frac{u(x_{i-2}, t_j) - 4u(x_{i-1}, t_j) + 3u(x_i, t_j)}{2h} - \bar{f}_j^i, \quad i = 2, \dots, N,\end{aligned} \tag{3b}$$

the residual of method $\mathcal{M}_s$. Here $\bar{f}_j^i = f(x_i, t_j, u(x_i, t_j), u_{t_j}(x_i, \cdot))$ is the value of the functional f calculated on the exact solution.

We will say that the residual has order $h^p + \Delta^q$ if there exists a constant C independent of Ψ_j^i, h and Δ such that $\|\Psi_j^i\| \le C(h^p + \Delta^q)$ for all $i = 1, \dots, N$, $j = 0, \dots, M$.

Theorem 1. *Let the exact solution $u(x, t)$ of problem (1) be thrice continuously differentiable with respect to state x, twice continuously differentiable with respect to time t and the first derivative of the solution with respect to x is continuously differentiable in t. Then the residual of method $\mathcal{M}_s$ has the order $h^2 + \Delta$.*

Proof. The residual is defined by (3). We expand the function $u(x, t)$ in a Taylor series in a neighborhood of the points (x_i, t_j) and (x_i, t_{j+1}), $i = 2, \dots, N$. We obtain the following equalities for the values of the function at these points:

$$\begin{aligned}
u(x_{i-1}, t_j) &= u(x_i, t_j) - h\frac{\partial u}{\partial x}(x_i, t_j) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}u(x_i, t_j) + O(h^3),\\
u(x_{i-2}, t_j) &= u(x_i, t_j) - 2h\frac{\partial u}{\partial x}(x_i, t_j) + \frac{4h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_i, t_j) + O(h^3),\\
u(x_{i-1}, t_{j+1}) &= u(x_i, t_{j+1}) - h\frac{\partial u}{\partial x}(x_i, t_{j+1}) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}u(x_i, t_{j+1}) + O(h^3),\\
u(x_{i-2}, t_{j+1}) &= u(x_i, t_{j+1}) - 2h\frac{\partial u}{\partial x}(x_i, t_{j+1}) + \frac{4h^2}{2}\frac{\partial^2 u}{\partial x^2}(x_i, t_{j+1}) + O(h^3),\\
u(x_i, t_{j+1}) &= u(x_i, t_j) + \frac{\partial u}{\partial t}(x_i, t_j)\Delta + O(\Delta^2).
\end{aligned}$$

Substituting these relations into (3) we obtain

$$\Psi_j^i = \frac{\partial u}{\partial t}(x_i, t_j) + O(\Delta) + as\left(\frac{\partial u}{\partial x}(x_i, t_{j+1}) + O(h^2)\right) + a(1-s)\left(\frac{\partial u}{\partial x}(x_i, t_j) + O(h^2)\right) - \bar{f}_j^i.$$

Now we expand the function $\frac{\partial u}{\partial x}(x, t)$ in a Taylor series in a neighborhood of the point (x_i, t_{j+1}), to obtain

$$\frac{\partial u}{\partial x}(x_i, t_{j+1}) = \frac{\partial u}{\partial x}u(x_i, t_j) + O(\Delta),$$

which yields

$$\Psi_j^i = \frac{\partial u}{\partial t}(x_i, t_j) + a\frac{\partial u}{\partial x}(x_i, t_j) + O(h^2) + O(\Delta) - \bar{f}_j^i.$$

Invoking (1a) we arrive at $\Psi_j^i = O(h^2 + \Delta)$. For $i = 1$ this theorem is proved in a similar way. □

Definition 3. Denote $\varepsilon_j^i = u(x_i, t_j) - u_j^i,\ i = 0, \ldots, N,\ j = 0, \ldots, M$. We say that method $\mathcal{M}_s$ converges if $\varepsilon_j^i \to 0$ when $h \to 0$ and $\Delta \to 0$ for all $i = 0, \ldots, N$ and $j = 0, \ldots, M$. We say that it converges with order $h^p + \Delta^q$, if there exists a constant C such that $\|\varepsilon_j^i\| \le C(h^p + \Delta^q)$ for all $i = 0, \ldots, N$ and $j = 0, \ldots, M$.

In the next section, we study problem of convergence and stability. The fundamental theorem in the analysis of finite difference methods for the numerical solution of partial differential equations without time delay is the Lax equivalence theorem. This theorem states [25] for a well-posed linear initial value problem, that a consistent finite difference method is convergent if and only if it is stable. In the case of equations with time delay one should deal with infinite-dimensional space, where it is difficult to build constructive and effective algorithms. If the difference scheme is finite-dimensional it must contain the delay term, therefore it is impossible to apply the Lax equivalence theorem directly. This problem was solved in [28] where the general difference scheme with aftereffect was elaborated. The principal modification was the introduction of an intermediate interpolation space.

In consideration of the nonlinear dependence of the functional f (and, consequently, F) on the state and its prehistory, the traditional methods of stability verification [24] are not applicable. However, to investigate the convergence of the schemes, as in the case of other evolutionary problems with delay effect, we can apply the technique of abstract schemes with aftereffect developed earlier [28] in the case of function-differential equations with ordinary derivatives. Below we describe the main points of this technique as applied to our case (see also [19]).

3. General difference scheme with aftereffect and its order of convergence

In this section, we reintroduce some of the notation used earlier, for example, τ and Δ. This is done deliberately for simplifying the embedding of the schemes from the previous section.

Let an interval $[t_0, \theta]$ be given, and let $\tau > 0$ be the value of the delay. Define the step of the grid $\Delta > 0$; to simplify the narration we assume that $\tau/\Delta = m$ and $(\theta - t_0)/\Delta = M$ are natural numbers. Denote by $\{\Delta\}$ the set of steps. A (uniform) grid is, by definition, a finite set of numbers $\Sigma_\Delta = \{t_i = t_0 + \Delta i,\ i = -m, \ldots, M\}$. We use the notation $\Sigma_\Delta^- = \{t_j \in \Sigma_\Delta,\ i < 0\}$ and $\Sigma_\Delta^+ = \{t_j \in \Sigma_\Delta,\ i \ge 0\}$.

A discrete model is defined as a grid function $t_i \in \Sigma_\Delta \mapsto y(t_i) = y_i \in Y,\ i = -m, \ldots, M$, where Y is a q-dimensional normed space with norm $\|\cdot\|_Y$. We will assume that the dimension q of the space Y depends on a number $h > 0$. The set $\{y_i\}_n = \{y_i \in Y,\ i = n - m, \ldots, n\}$ will be called the prehistory of the discrete model by the time t_n, $0 \le n \le M$. Let V be a linear normed space with norm $\|\cdot\|_V$, so-called interpolation space. A mapping $\{y_i\}_n \mapsto I(\{y_i\}_n) = v \in V$ is, by definition, an operator of the interpolation of the discrete prehistory.

We will say that the interpolation operator satisfies the Lipschitz condition if there exists a constant L_I such that, for all prehistories $\{y_i^1\}_n$ and $\{y_i^2\}_n$ of the discrete model, the following inequality holds:

$$\left\|I\left(\{y_i^1\}_n\right) - I\left(\{y_i^2\}_n\right)\right\|_V \le L_I \max_{n-m\le i\le n} \left\|y_n^1 - y_n^2\right\|_Y. \tag{4}$$

Starting values of the model are defined by the function acting from Σ_Δ^- to Y:

$$y(t_i) = y_i, \quad i = -m, \ldots, 0. \tag{5}$$

The formula of the advance of the model by a step is, by definition, the relation

$$y_{n+1} = S(y_n) + \Delta\,\Phi(t_n, I(\{y_i\}_n), \Delta), \quad n = 0, \ldots, M-1, \tag{6}$$

where $\Phi : \Sigma_\Delta^+ \times V \times \{\Delta\} \to Y$ is the function of advance by a step and the transition operator $S : Y \to Y$ is a linear operator.

Thus, a discrete model (in what follows, simply a method) is defined by starting values (5), formula of advance by a step (6), and an interpolation operator. We assume that the function $\Phi(t_n, v, \Delta)$ in (6) is Lipschitz with respect to the second argument; i.e., there exists a constant L_Φ such that, for all $t_n \in \Sigma_\Delta^+$, $\Delta \in \{\Delta\}$, and $v^1, v^2 \in V_n$ the following inequality holds:

$$\left\|\Phi(t_n, v^1, \Delta) - \Phi(t_n, v^2, \Delta)\right\|_Y \le L_\Phi \left\|v^1 - v^2\right\|_V.$$

The function of exact values is, by definition, the mapping $Z(t_i, \Delta) = z_i \in Y,\ i = -m, \ldots, M$. We will say that starting values of the model have order $\Delta^{p_1} + h^{p_2}$ if there exists a constant C independent of z_i, y_i, Δ, and h such that

$$\|z_i - y_i\|_Y \le C\left(\Delta^{p_1} + h^{p_2}\right), \quad i = -m, \ldots, 0.$$

We will say that the method converges with order $\Delta^{p_1} + h^{p_2}$ if there exists a constant C independent of z_i, y_i, Δ, and h such that for all $n = -m, \dots, M$, the following inequality holds:

$$\|z_n - y_n\|_Y \le C\left(\Delta^{p_1} + h^{p_2}\right).$$

In what follows, we will omit subscripts at norms. Method (6) is called stable if $\|S\| \le 1$. An error of approximation with interpolation (a residual) is, by definition, the grid function

$$d_n = (z_{n+1} - S(z_n))/\Delta - \Phi(t_n, I(\{z_i\}_n), \Delta), \quad n = 0, \dots, M-1. \tag{7}$$

We will say that method (6) has order of error of approximation with interpolation $\Delta^{p_1} + h^{p_2}$ if there exists a constant C independent of d_n, Δ, and h such that for all $n = 1, \dots, M$, the following inequality holds:

$$\|d_n\| \le C(\Delta^{p_1} + h^{p_2}).$$

Theorem 2. *Suppose that method (6) is stable, the function Φ satisfies the Lipschitz condition with respect to the second argument, the interpolation operator I satisfies the Lipschitz condition, the starting values have order $\Delta^{p_1} + h^{p_2}$, and the error of approximation with interpolation has order $\Delta^{p_3} + h^{p_4}$, where $p_i > 0$, $i = 1, \dots, 4$. Then, the method converges and the order of the convergence is at least $\Delta^{\min\{p_1,p_3\}} + h^{\min\{p_2,p_4\}}$.*

Proof. Let $\delta_n = z_n - y_n$ for $n = -m, \dots, M$, then, for $n = 0, \dots, M-1$ we have

$$\delta_{n+1} = S(\delta_n) + \Delta\,\hat{\delta}_n + \Delta\, d_n, \tag{8}$$

where $\hat{\delta}_n = \Phi(t_n, I(\{z_i\}_n), \Delta) - \Phi(t_n, I(\{y_i\}_n), \Delta)$. The assumptions that the mappings Φ and I are Lipschitz imply that

$$\|\hat{\delta}_n\| \le K \max_{n-m\le i\le n}\{\|\delta_i\|\}, \tag{9}$$

where $K = L_I L_\Phi$. It follows from (8) that

$$\delta_{n+1} = S^{n+1}(\delta_0) + \Delta\sum_{j=0}^{n} S^{n-j}(\hat{\delta}_j) + \Delta\sum_{j=0}^{n} S^{n-j}(d_j). \tag{10}$$

From (9) and (10) together with the stability of S we have

$$\|\delta_{n+1}\| \le \|\delta_0\| + K\Delta\sum_{j=0}^{n}\max_{j-m\le i\le j}\|\delta_i\| + (\theta - t_0)\max_{0\le i\le N-1}\|d_i\|. \tag{11}$$

Let us denote $R_0 = \max_{-m\le i\le 0}\|\delta_i\|$, $R = \max_{0\le i\le N-1}\|d_i\|$, and $D = R_0 + (\theta - t_0)R$ so we can rewrite estimate (11) in the form

$$\|\delta_{n+1}\| \le K\Delta\sum_{j=0}^{n}\max_{j-m\le i\le j}\|\delta_i\| + D. \tag{12}$$

Suppose the following estimate

$$\|\delta_n\| \le D(1 + K\Delta)^n, \tag{13}$$

is valid for all $n = 1, \dots, M$. From this we obtain $\|\delta_{n+1}\| \le D\,\exp(K(\theta - t_0))$, which implies the conclusion of the theorem, as by definition of D, the inequality $D < C\left(\Delta^{\min\{p_1,p_3\}} + h^{\min\{p_2,p_4\}}\right)$ holds. It remains to prove (13). We proceed by induction.

Induction base. If we set $n = 0$ in (12), then $\|\delta_1\| \le K\Delta\|\delta_0\| + D \le (1 + K\Delta)D$.

Induction step. Let estimate (13) be valid for all indices from 1 to n. Let us show that the estimate is also valid for $n + 1$. Fix $j \le n$, and let $i_0 = i_0(j)$ be an index for which $\max_{j-m\le i\le j}\|\delta_i\|$ is attained. The following two situations are possible:

- $i_0 \le 0$, then, $\max_{j-m\le i\le j}\|\delta_i\| = \|\delta_{i_0}\| \le R_0 \le D(1 + K\Delta)^j$;
- $1 \le i_0 \le j$, then, by the induction hypothesis

$$\max_{j-m\le i\le j}\|\delta_i\| = \|\delta_{i_0}\| \le D(1 + K\Delta)^{i_0} \le D(1 + K\Delta)^j.$$

Thus, the following estimate is valid in any case $\max_{j-m\le i\le j}\|\delta_i\| \le D(1 + K\Delta)^j$. This estimate combined with (12) yields

$$\|\delta_{n+1}\| \le K\Delta\sum_{j=0}^{n} D(1 + K\Delta)^j + D = D(1 + K\Delta)^{n+1},$$

by which (13) is proved. $\square$

4. Stability and convergence order of the scheme $\mathcal{M}_s$

In this section we consider problems with the homogeneous boundary condition $u(0,t) = 0,\ t \in [t_0, \theta]$. The replacement $\tilde{u}(x,t) = u(x,t) - g(t)$ turns the initial problem into the mentioned one. Let us embed the schemes from family $\mathcal{M}_s$ into the general difference scheme with aftereffect.

For every $t_j \in \Sigma_\Delta$ define the values of the discrete model by the vector $\mathbf{y}_j = \left(u_j^1, u_j^2, \ldots, u_j^N\right)^\top \in Y,\ j = 0, \ldots, M-1$, where the sign $\cdot^\top$ means transposition, Y is a linear space. On the space Y we define a linear operator A by the matrix

$$\mathsf{A} = \frac{a}{2h}\begin{pmatrix} 4 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -4 & 3 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -4 & 3 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & -4 & 3 & \cdots & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & \cdots & 1 & -4 & 3 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -4 & 3 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & -4 & 3 \end{pmatrix}.$$

Then we can rewrite system $\mathcal{M}_s$ in the form

$$\frac{\mathbf{y}_{j+1} - \mathbf{y}_j}{\Delta} + s\mathsf{A}\,\mathbf{y}_{j+1} + (1-s)\mathsf{A}\,\mathbf{y}_j = \mathbf{F}_j, \tag{14}$$

here $\mathbf{F}_j = \left(f_j^1 + s f_{j+1}^0 + (1-s) f_j^0,\ f_j^2, \ldots, f_j^N\right)^\top$. Let us use the obvious identity

$$\mathbf{y}_{j+1} = \mathbf{y}_j + \Delta\frac{\mathbf{y}_{j+1} - \mathbf{y}_j}{\Delta}$$

and define the linear operator $B = E + s\Delta A$, (E is the identity operator) to rewrite (14) as a two-layer difference scheme in the canonical form [24]

$$\mathsf{B}\frac{\mathbf{y}_{j+1} - \mathbf{y}_j}{\Delta} + \mathsf{A}\,\mathbf{y}_j = \mathbf{F}_j. \tag{15}$$

The operator A is positive definite with eigenvalues $\lambda_1(A) = 2a/h,\ \lambda_2(A) = \cdots = \lambda_n(A) = 3a/2h$, therefore B is a positive definite operator. Since B is invertible, we can rewrite (15) in the form

$$\mathbf{y}_{j+1} = \mathsf{S}\,\mathbf{y}_j + \Delta\mathsf{B}^{-1}\,\mathbf{F}_j,$$

where $S = (E - \Delta B^{-1} A)$ is the transition operator.

In the space Y we introduce a scalar product and the energy norm

$$(y, u) = \sum_{i=1}^{N} y^i u^i h, \qquad \|y\|_Y = \sqrt{(Ay, y)},$$

thereafter we define the corresponding induced operator norm.

Definition 4. The difference scheme (15) is said to be stable, if $\|S\|_Y < 1$.

Note that the equivalent formalization of stability of two-layer difference schemes is given in [24, pp. 324–330].

Theorem 3. *If the condition $s \geq 1/2$ is fulfilled then the difference scheme (15) is stable.*

Proof. Let us consider (15) from the point of view of operator-difference equations and apply methods of the stability verification for a two-layer difference scheme [24] and the separation of finite-dimensional and infinite-dimensional components [14,19].

We symmetrize (15) by multiplying through by A^{-1}:

$$(\mathsf{A}^{-1} + s\Delta\mathsf{E})\frac{\mathbf{y}_{j+1} - \mathbf{y}_j}{\Delta} + \mathsf{E}\,\mathbf{y}_j = \mathsf{A}^{-1}\mathbf{F}_j.$$

Denoting $\hat{B} = A^{-1} + s\Delta E,\ \hat{A} = E$, and $\hat{\mathbf{F}}_j = \mathsf{A}^{-1}\,\mathbf{F}_j$, we obtain

$$\hat{\mathsf{B}}\,\frac{\mathbf{y}_{j+1} - \mathbf{y}_j}{\Delta} + \hat{\mathsf{A}}\,\mathbf{y}_j = \hat{\mathbf{F}}_j. \tag{16}$$

Method (16) is stable in the energy norm if and only if $2\hat{B} \geq \hat{A}$, see [24, p. 333 Theorem 1]. This requirement is equivalent to $A^{-1} + \Delta E(s - 0.5) \geq 0$. Since A^{-1} is a positive definite operator, the latter inequality is fulfilled for any Δ, as soon as $s \geq 0.5$. $\square$

Table 1
Numerical results of Experiment 1.

Case	1	2	3	4	5	6	7	8
h	1/5	1/10	1/10	1/20	1/5	1/10	1/20	1/40
Δ	$\pi/20$	$\pi/20$	$\pi/40$	$\pi/40$	$\pi/200$	$\pi/200$	$\pi/200$	$\pi/200$
`diff`	0.6511	0.7045	0.4652	0.5082	0.2534	0.0647	0.0180	0.0165
CPU-time	0.011	0.019	0.036	0.067	0.095	0.164	0.305	0.581

The peculiar feature of the presented method is the condition $s \geq 1/2$ does not impose any restriction on the step size like the Courant–Friedrichs–Lewy condition does. Note that in difference schemes for parabolic and hyperbolic equations with time delay [20,19,22,21] such conditions of Courant type are essential.

We define the function of exact values by the relations

$$\mathbf{z}_j = (u(x_1, t_j), u(x_2, t_j), \dots, u(x_N, t_j))^\top \in Y.$$

Starting values of the model can be taken equal to the function of exact values

$$\mathbf{y}_j = \mathbf{z}_j = (\varphi(x_1, t_j), \varphi(x_2, t_j), \dots, \varphi(x_N, t_j))^\top, \quad j = -m, \dots, 0.$$

The definition of the residual without interpolation (3) in the scheme with weights for the equation advection with time delay and retardation of a state variable and the definition of the residual with interpolation (7) in the general scheme are essentially different. However, the following obvious statement connects these two definitions.

Theorem 4. *Let the conditions of Theorem 3 be satisfied and the interpolation operator (2) is used. Then, the residual with interpolation in the sense of (7) has order $h^2 + \Delta$.*

The embedding of the scheme with weights for Eq. (1a) into the general scheme has been carried out, thereafter the following statement is true.

Theorem 5. *Let the exact solution $u(x, t)$ of problem (1) be thrice continuously differentiable with respect to state x, twice continuously differentiable with respect to time t and the first derivative of the solution with respect to x is continuously differentiable in t. Then if $2s > 1$ method $\mathcal{M}_s$ converges with order $h^2 + \Delta$.*

5. Numerical experiments

Simulations were done in MATLAB [29], on a PC ASUS, CPU Intel Core i5-2401M, 2.3 GHz, 4 Gb RAM.

Experiment 1. Let us consider the following first order partial differential equation with discrete time delay $\tau = \pi/2$ and retardation of a state variable with biasing constant $\alpha = 1/2$:

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial t} = u(x, t) - e^{x/2} u\left(x/2, t - \pi/2\right) + e^x \sin t, \quad 0 < x < 2,\ 0 < t \leq 2\pi,$$

with initial and boundary conditions

$$u(x, t) = e^x \sin t, \quad 0 \leq x \leq 2, \quad -\pi/2 \leq t \leq 0,$$
$$u(0, t) = \sin t, \quad 0 \leq t \leq 2\pi.$$

This boundary value problem has the exact solution $u(x, t) = e^x \sin t$. In Table 1 we report the deviations $\mathtt{diff} = \max_{i,j} |u^i_j - u(x_i, t_j)|$ of the approximate solution calculated by method $\mathcal{M}_s$ with $s = 0.8$ from the exact one for different values of h and Δ. We also report the CPU-time.

In cases nos. 5–7 the error related to the time discretization is small in comparison with the error related to the spatial discretization; the analysis of the error behavior reveals the square convergence with respect to x, i.e., when the step becomes half as much, the error becomes almost four times less. The analysis of the data in the table shows that only the consistent decrease of steps yields the decrease of error. Thus, in cases nos. 7–8 the halving of h does not cause the corresponding decrease of error, because the total error is mostly induced by the time discretization.

By Theorem 3 for $s = 0.8$ scheme $\mathcal{M}_s$ is stable with any ratio of steps; however, due to the ill-posedness of the numerical differentiation, the decrease of h makes the approximation of $\partial u/\partial x$ in $\mathcal{M}_s$ more sensitive to the computer rounding error, which leads to the increase of the error. The decrease of Δ consistent with h is a peculiar regularizer which prevents errors from growing and accumulating. Cases nos. 1–4 illustrate this fact.

Experiment 2. Let us consider the following first order partial differential equation with discrete and distributed time delay $\tau = \pi$, and retardation of a state variable with biasing constant $\alpha = 1/3$:

$$\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial t} = -u(x, t)^2 - 2u\left(x/3, t - \pi/2\right) + \int_0^\pi u\left(x/3, t - \xi\right) d\xi, \quad 0 < x < 2\pi,\ \pi < t \leq 4\pi,$$

Table 2
Numerical results of Experiment 2.

Case	1	2	3	4	5	6	7	8
h	$\pi/10$	$\pi/20$	$\pi/20$	$\pi/40$	$\pi/10$	$\pi/20$	$\pi/40$	$\pi/80$
Δ	$\pi/20$	$\pi/20$	$\pi/40$	$\pi/40$	$\pi/400$	$\pi/400$	$\pi/400$	$\pi/400$
`diff`	0.0808	0.0826	0.0421	0.0425	0.0420	0.0131	0.0054	0.0041
CPU-time	0.258	0.565	1.07	2.17	2.55	5.35	10.93	22.45

with initial and boundary conditions

$$u(x,t) = \cos(x+t), \quad 0 \le x \le 2\pi,\ 0 \le t \le \pi,$$
$$u(0,t) = \cos t, \quad \pi \le t \le 4\pi.$$

This boundary value problem has the exact solution $u(x,t) = \cos(x+t)$.

The results of this numerical simulation are represented in Table 2 for parameter $s = 0.8$. Remark that the order of numerical integration must be consistent with the order of difference method $\mathcal{M}_s$.

6. Conclusion

We considered a first order partial differential equation with a time delay and retardation of a state variable for which we constructed a finite difference scheme for its numerical solution. The order of approximation, stability and convergence order of the numerical schemes are given. We demonstrated the obtained results on two examples.

Acknowledgments

This research is supported by RFBR 14-01-00065 and 13-01-00089. We acknowledge the support by the program 02.A03.21.0006 on 27.08.2013.

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